

Evolution of a dense neutrino gas in matter and electromagnetic field

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(ΩDated: November 1, 2011)

We describe the system of massive Weyl fields propagating in a background matter and interacting with an external electromagnetic field. The interaction with an electromagnetic field is due to the presence of anomalous magnetic moments. To canonically quantize this system first we develop the classical field theory treatment of Weyl spinors in frames of the Hamilton formalism which accounts for the external fields. Then, on the basis of the exact solution of the wave equation for a massive Weyl field in a background matter we obtain the effective Hamiltonian for the description of spin-flavor oscillations of Majorana neutrinos in matter and a magnetic field. Finally, we incorporate in our analysis the neutrino self-interaction which is essential when the neutrino density is sufficiently high. We also discuss the applicability of our results for the studies of collective effects in spin-flavor oscillations of supernova neutrinos in a dense matter and a strong magnetic field.

Keywords: Weyl field, background matter, electromagnetic field, Majorana neutrino, spin-flavor oscillations, collective effects

I. INTRODUCTION

It is known that the neutrino interaction with external fields can significantly influence the evolution of supernova neutrinos [1]. For example, the combined action of a background matter and a magnetic field can cause the resonant transition like, $\nu_{\alpha}^{-} \leftrightarrow \nu_{\beta}^{+}$ (see, e.g., Ref. [2]), where the indexes \pm denote different helicity states. Hence the active neutrinos of the flavor α can be converted into sterile neutrinos of another flavor β .

Besides external fields, other factors, like the neutrino self-interaction, can strongly influence neutrino oscillations. It happens, e.g., in a supernova explosion, when the typical neutrino luminosity can be $\sim 10^{51}$ erg/s [3]. Accounting for the average supernova neutrino energy $E_{\nu} \sim 10$ MeV, we get that at the distance about several tens of kilometers from the protoneutron star surface the neutrino number density still can be high enough for interactions between neutrinos to be as important as the neutrino interaction with external fields. This neutrino self-interaction leads to the collective effects in neutrino oscillations.

For the first time the neutrino self-interaction was considered in Ref. [4]. Since then a lot of works on this subject has been published (see, e.g., the recent review [5] and references therein). Note that in the majority of the studies of collective effects in neutrino oscillations only the combination of the interaction with a background matter and the neutrino self-interaction was considered (see, e.g., Ref. [6]). In the present work we shall generalize the previous approaches for the description of collective neutrino oscillations to include the interaction with

an external electromagnetic field since, as we mentioned above, the influence of a strong magnetic field on the neutrino system evolution can be also important.

Neutrinos can interact with an external electromagnetic field due to the presence of anomalous magnetic moments. Note that the structure of the magnetic moments is completely different for Dirac and Majorana neutrinos (see, e.g., Ref. [7]). Despite the fact that nowadays there is no universally recognized confirmation of the nature of neutrinos [8], in the present work we shall suppose that neutrinos are Majorana particles. Note that in various scenarios for the generation of elementary particles masses, it is predicted that neutrinos should acquire Majorana masses [9].

In the present work we shall resolve several important problems for the physics of Majorana neutrinos interacting with external fields. First, in Sec. II, using the results of our recent paper [10] we propose the classical field theory treatment of massive Weyl fields propagating in background matter and interacting with an external electromagnetic field. Then, in Sec. III, on the basis of the exact solution of the wave equation for Weyl fields in a background matter we canonically quantize these fields. In Sec. IV, in frames of our method we re-derive the effective Hamiltonian for the description of spin-flavor oscillations of Majorana neutrinos in matter and a magnetic field. Finally, in Sec. V, we apply the developed formalism to get the contribution of the neutrino self-interaction to the effective Hamiltonian. In Sec. VI, we summarize our results.

II. CLASSICAL FIELD THEORY

In this section we develop the classical field theory description of the massive neutrinos eigenstates, which are

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supposed to be Majorana particles, in matter and electromagnetic field. For this purpose we derive the Hamiltonian for the system of two-component Weyl spinors and show that the classical canonical equations are equivalent to the wave equation for a Majorana neutrino in external fields.

The wave equation for the neutrino mass eigenstates ψ_a , propagating in background matter and interacting with an external electromagnetic field, is known to have the form,

$$(i\gamma^\mu \partial_\mu - m_a)\psi_a - \frac{\mu_{ab}}{2}\sigma_{\mu\nu}F^{\mu\nu}\psi_b + g_{ab}^\mu\gamma_\mu\gamma^5\psi_b = 0, \quad (2.1)$$

where m_a are the masses of the particles, γ^μ , γ^5 , and $\sigma_{\mu\nu} = (i/2)(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$ are the Dirac matrices. Note that we will formulate the dynamics of the system (2.1) in the mass eigenstates basis rather than in the flavor basis, as it is usually done when neutrino oscillations are considered, since only in the mass eigenstates basis one can distinguish between Dirac and Majorana masses [11].

The interaction with matter is characterized by the external fields (g_{ab}^μ), which, in principle, are nondiagonal in the neutrino mass eigenstates basis. In general case the matrix (g_{ab}^μ) is hermitian. However we shall discuss the situation when the CP invariance is conserved. Despite a current attempt to detect CP violating terms in the neutrino sector [12], no definite results have been obtained yet. In this case the vacuum mixing matrix is orthogonal and the matrix (g_{ab}^μ) is symmetric. The zero component of this matrix, (g_{ab}^0), contains the effective potentials of the neutrino interaction with non-moving and unpolarized matter, whereas the vector components, (\mathbf{g}_{ab}), are the linear combinations of the averaged matter velocity and the polarization. The explicit form of these matrices and the details of the statistical averaging can be found in Ref. [13].

Note that the vector term in the neutrino matter interaction $\sim g_{ab}^\mu\gamma_\mu\psi_b$ is omitted in Eq. (2.1) since it is washed out for Majorana neutrinos. The contribution of the axial-vector interaction with matter to the wave equation (2.1) $\sim g_{ab}^\mu\gamma_\mu\gamma^5\psi_b$ is twice the analogous contribution for Dirac particles since both neutrinos and antineutrinos equally interact with a background matter (see, e.g., Ref. [14]).

Neutrinos can interact with an external electromagnetic field $F_{\mu\nu} = (\mathbf{E}, \mathbf{B})$ owing to the presence of the

anomalous magnetic moments (μ_{ab}). It is known (see, e.g., Ref. [15]) that the matrix (μ_{ab}) should be hermitian and pure imaginary, i.e. $\mu_{ab} = -\mu_{ba}^*$ and $\mu_{ab}^* = -\mu_{ab}$. We shall discuss the situation when no admixture of sterile neutrinos is in the mass eigenstates ψ_a . In this case the electric dipole moments are equal to zero [15].

We define the interaction with external fields in the mass eigenstates basis. However the interaction with a background matter is usually given for flavor neutrinos (see, e.g., Ref. [2]). For the detailed discussion of the explicit forms of the matrices (g_{ab}^μ) and (μ_{ab}) in the flavor eigenstates basis the reader is referred to the recent review [16].

Since the neutrino mass eigenstates ψ_a are supposed to be Majorana particles they should obey the Majorana condition in the form, $\psi_a^c = i\gamma^2\psi_a^* = \kappa_c\psi_a$, where κ_c is a phase factor which we shall take equal to one. If we express the four-component Majorana spinors in terms of two-component Weyl fields, η_a and ξ_a , as

$$\psi_a^{(\eta)} = \begin{pmatrix} i\sigma_2\eta_a^* \\ \eta_a \end{pmatrix}, \quad \text{or} \quad \psi_a^{(\xi)} = \begin{pmatrix} \xi_a \\ -i\sigma_2\xi_a^* \end{pmatrix}, \quad (2.2)$$

which satisfy the Majorana condition, we can rewrite Eq. (2.1) in the two equivalent forms,

$$\begin{aligned} i\dot{\eta}_a - (\boldsymbol{\sigma}\nabla)\eta_a + m_a\sigma_2\eta_a^* - \mu_{ab}\boldsymbol{\sigma}(\mathbf{B} - i\mathbf{E})\sigma_2\eta_b^* \\ + i(g_{ab}^0 + \boldsymbol{\sigma}\mathbf{g}_{ab})\eta_b = 0, \end{aligned} \quad (2.3)$$

or

$$\begin{aligned} i\dot{\xi}_a + (\boldsymbol{\sigma}\nabla)\xi_a - m_a\sigma_2\xi_a^* + \mu_{ab}\boldsymbol{\sigma}(\mathbf{B} + i\mathbf{E})\sigma_2\xi_b^* \\ - i(g_{ab}^0 - \boldsymbol{\sigma}^*\mathbf{g}_{ab})\xi_b = 0. \end{aligned} \quad (2.4)$$

Here $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices. In the following we shall postulate these equations. Note that the analog of Eq. (2.3) was previously derived in Ref. [17]. We shall choose the spinors η_a as the basic ones since it was experimentally confirmed that active neutrinos correspond to left-handed particles. Note that the rigorous proof of the equivalence of Majorana and Weyl fields was given in Ref. [18].

In Ref. [10] we demonstrated that the classical dynamics of a massive Weyl field in vacuum should be described only in frames of the Hamilton formalism. Generalizing the results of Ref. [10] to include the interaction with a background matter and an electromagnetic field we arrive to the following Hamiltonian:

$$\begin{aligned} H = \int d^3\mathbf{r} \left[\sum_a \left\{ \pi_a^T (\boldsymbol{\sigma}\nabla)\eta_a - (\eta_a^*)^T (\boldsymbol{\sigma}\nabla)\pi_a^* + m_a [(\eta_a^*)^T \sigma_2 \pi_a + (\pi_a^*)^T \sigma_2 \eta_a] \right\} \right. \\ \left. + \sum_{ab} \left\{ \mu_{ab} [\pi_a^T \boldsymbol{\sigma}(\mathbf{B} - i\mathbf{E})\sigma_2\eta_b^* + \eta_a^T \sigma_2 \boldsymbol{\sigma}(\mathbf{B} + i\mathbf{E})\pi_b^*] - i[\pi_a^T (g_{ab}^0 + \boldsymbol{\sigma}\mathbf{g}_{ab})\eta_b - (\eta_a^*)^T (g_{ab}^0 - \boldsymbol{\sigma}^*\mathbf{g}_{ab})\pi_b^*] \right\} \right], \end{aligned} \quad (2.5)$$

where π_a are the canonical momenta conjugate to the “coordinates” η_a . Using the aforementioned properties of the matrices (μ_{ab}) and (g_{ab}^μ) we find that the functional (2.5) is real as it should be for a classical Hamiltonian.

Applying the field theory version of the canonical equations to the Hamiltonian H ,

$$\dot{\eta}_a = \frac{\delta H}{\delta \pi_a} = (\boldsymbol{\sigma} \nabla) \eta_a - m_a \sigma_2 \eta_a^* + \mu_{ab} \boldsymbol{\sigma} (\mathbf{B} - i\mathbf{E}) \sigma_2 \eta_b^* - i(g_{ab}^0 + \boldsymbol{\sigma} \mathbf{g}_{ab}) \eta_b, \quad (2.6)$$

$$\dot{\pi}_a = -\frac{\delta H}{\delta \eta_a} = (\boldsymbol{\sigma}^* \nabla) \pi_a + m_a \sigma_2 \pi_a^* - \mu_{ab} \sigma_2 \boldsymbol{\sigma} (\mathbf{B} + i\mathbf{E}) \pi_b^* + i(g_{ab}^0 + \boldsymbol{\sigma}^* \mathbf{g}_{ab}) \pi_b, \quad (2.7)$$

one can see that in Eq. (2.6) we reproduce Eq. (2.3) for Weyl particles, which correspond to left-handed neutrinos, interacting with matter and electromagnetic field. If we introduce the new variable $\xi_a = i\sigma_2 \pi_a$, we can show that Eq. (2.7) is equivalent to Eq. (2.4) for right-handed neutrinos.

Previous quantum field theory based studies of Majorana neutrinos in an electromagnetic field and in a background matter [20, 21] involved the Lagrange formalism. The mass term in a Lagrangian for the Weyl field η_a has the form (see, e.g., Ref. [18]),

$$\mathcal{L}_m = -\frac{i}{2} m_a \eta_a^T \sigma_2 \eta_a + \frac{i}{2} m_a \eta_a^\dagger \sigma_2 \eta_a^*. \quad (2.8)$$

It is however clear that \mathcal{L}_m vanishes if η_a is a first-quantized field having commuting c -number components. To resolve this problem in Refs. [20, 21] it was supposed that two-component Weyl fields are represented via the anti-commuting operators. Thus the treatment of Majorana particles in those works can be considered just as the re-expression of already quantized fields in terms of the new variables rather than the generic canonical quantization. Moreover in Ref. [20] it was claimed that “there is no ‘first-quantized’ description of a massive two-component field in terms of c -number wave functions”.

On the contrary, in the present work we have demonstrated that classical (first-quantized) massive Weyl fields in presence of an external electromagnetic field and a background matter can be perfectly described within the Hamilton formalism, cf. Eqs. (2.5)-(2.7). This fact just means that the Lagrange formalism is not a suitable tool for the studies of Majorana particles. Note that a more detailed description of the Weyl fields dynamics in vacuum in frames of the canonical approach as well as the discussion of the applicability of the Lagrange and the Hamilton formalisms is presented in our recent work [10].

III. QUANTIZATION

In this section we canonically quantize a Weyl field propagating in a background matter. On the basis of an exact solution of the wave equation for a massive Weyl field in matter we express the energy and the momentum of the field as a sum of contributions of independent quantum oscillators. From the requirement of the positive definiteness of the energy it turns out that the oper-

ators in the decomposition of the wave functions should obey the Fermi-Dirac statistics.

In the following we shall suppose that the background matter in average is at rest and unpolarized, i.e. $\mathbf{g}_{ab} = 0$. This approximation is valid in almost all realistic cases. Indeed the matter motion is relevant for the neutrino dynamics if the speed of medium is comparable with the speed of light. This situation may be implemented, e.g., if a beam of neutrinos propagates inside a relativistic jet from a quasar. Since the results of the present work are likely to be applied for the studies of supernova neutrinos, the matter motion seems to be irrelevant for us.

When neutrinos interact with a non-degenerate plasma, we may neglect the matter polarization if $\mu_f |\mathbf{B}| \ll T$ (see Ref. [19]), where $\mu_f \sim \mu_B$ is the magnetic moment of a background fermion, μ_B is the Bohr magneton, and T is the plasma temperature. In the present work we shall study the influence of both external fields (see Sec. IV) and the neutrino self-interaction (see Sec. V) on neutrino oscillations. In Ref. [22] it was found that the collective effects in neutrino oscillations reveal themselves most intensively at the distance $r \sim 100$ km from a protoneutron star. The magnetic field at this distance can be $B = B_0 (R/r)^3 \sim 10^{10}$ G, where $B_0 \sim 10^{13}$ G is the typical magnetic field on the protoneutron star surface and $R \sim 10$ km is the stellar radius. Supposing that $T \sim 1$ MeV [23], we get that the matter polarization becomes unimportant for stars possessing moderate magnetic fields.

Let us decompose the Hamiltonian (2.5) into two terms $H = H_0 + H_{\text{int}}$. The former term, H_0 , contains the vacuum Hamiltonian as well as the matter interaction term diagonal in the neutrino types,

$$H_0 = \int d^3 \mathbf{r} \sum_a \left\{ \pi_a^T [(\boldsymbol{\sigma} \nabla) - i g_{aa}^0] \eta_a - (\eta_a^*)^T [(\boldsymbol{\sigma} \nabla) - i g_{aa}^0] \pi_a^* + m_a [(\eta_a^*)^T \sigma_2 \pi_a + (\pi_a^*)^T \sigma_2 \eta_a] \right\}. \quad (3.1)$$

The latter term in this decomposition,

$$H_{\text{int}} = \int d^3 \mathbf{r} \sum_{a \neq b} \left\{ \mu_{ab} [\pi_a^T \boldsymbol{\sigma} (\mathbf{B} - i\mathbf{E}) \sigma_2 \eta_b^* + \eta_a^T \sigma_2 \boldsymbol{\sigma} (\mathbf{B} + i\mathbf{E}) \pi_b^*] - i g_{ab}^0 [\pi_a^T \eta_b - (\eta_a^*)^T \pi_b^*] \right\}, \quad (3.2)$$

has the nondiagonal matter interaction and the interaction with an electromagnetic field which is nondiagonal by definition.

Analogously to Eqs. (2.6) and (2.7) we define the reduced Hamilton equations which contain only the Hamiltonian H_0 : $\dot{\eta}_a^{(0)} = \delta H_0 / \delta \pi_a^{(0)}$ and $\dot{\pi}_a^{(0)} = -\delta H_0 / \delta \eta_a^{(0)}$. Using the results of Refs. [10, 17] we can find the solutions of these equations in the form,

$$\begin{aligned}
\eta_a^{(0)}(\mathbf{r}, t) &= \int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}} \left\{ \left[a_a^- w_- e^{-iE_a^- t} - \frac{m_a}{E_a^+ + |\mathbf{p}| - g_{aa}^0} a_a^+ w_+ e^{-iE_a^+ t} \right] e^{i\mathbf{p}\mathbf{r}} \right. \\
&\quad \left. + \left[(a_a^+)^* w_- e^{iE_a^+ t} + \frac{m_a}{E_a^- + |\mathbf{p}| + g_{aa}^0} (a_a^-)^* w_+ e^{iE_a^- t} \right] e^{-i\mathbf{p}\mathbf{r}} \right\}, \\
\xi_a^{(0)}(\mathbf{r}, t) &= i \int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}} \left\{ \left[b_a^+ w_+ e^{-iE_a^+ t} + \frac{m_a}{E_a^- + |\mathbf{p}| + g_{aa}^0} b_a^- w_- e^{-iE_a^- t} \right] e^{i\mathbf{p}\mathbf{r}} \right. \\
&\quad \left. + \left[(b_a^-)^* w_+ e^{iE_a^- t} - \frac{m_a}{E_a^+ + |\mathbf{p}| - g_{aa}^0} (b_a^+)^* w_- e^{iE_a^+ t} \right] e^{-i\mathbf{p}\mathbf{r}} \right\}, \tag{3.3}
\end{aligned}$$

where we introduce the new variable $\xi_a^{(0)} = i\sigma_2 \pi_a^{(0)}$, w_{\pm} are the helicity amplitudes defined in Ref. [24], and

$$E_a^{(\zeta)} = \sqrt{m_a^2 + (|\mathbf{p}| - \zeta g_{aa}^0)^2}, \tag{3.4}$$

is the energy of a Weyl field [14, 21], $\zeta = \pm 1$ is the particle helicity. To derive Eqs. (3.3) and (3.4) we suppose that the external field g_{aa}^0 is spatially constant.

Using Eq. (3.3) we can express the Hamiltonian (3.1) as

$$\begin{aligned}
H_0 &= \int d^3\mathbf{p} \sum_a \left\{ E_a^- [a_a^-(\mathbf{p})]^* b_a^-(\mathbf{p}) + [b_a^-(\mathbf{p})]^* a_a^-(\mathbf{p}) \right. \\
&\quad + \frac{m_a^2}{(E_a^- + |\mathbf{p}| + g_{aa}^0)^2} \{ a_a^-(\mathbf{p}) [b_a^-(\mathbf{p})]^* + b_a^-(\mathbf{p}) [a_a^-(\mathbf{p})]^* \} \\
&\quad - E_a^+ [a_a^+(\mathbf{p})] b_a^+(\mathbf{p}) + b_a^+(\mathbf{p}) [a_a^+(\mathbf{p})]^* \\
&\quad + \frac{m_a^2}{(E_a^+ + |\mathbf{p}| - g_{aa}^0)^2} \{ [b_a^+(\mathbf{p})]^* a_a^+(\mathbf{p}) + [a_a^+(\mathbf{p})]^* b_a^+(\mathbf{p}) \} \\
&\quad + i m_a \left\{ \frac{E_a^-}{E_a^- + |\mathbf{p}| + g_{aa}^0} [e^{-2iE_a^- t} \{ a_a^-(\mathbf{p}) b_a^-(\mathbf{p}) + b_a^-(\mathbf{p}) a_a^-(\mathbf{p}) \} \right. \\
&\quad + e^{2iE_a^- t} \{ [a_a^-(\mathbf{p})]^* [b_a^-(\mathbf{p})]^* + [b_a^-(\mathbf{p})]^* [a_a^-(\mathbf{p})]^* \}] \\
&\quad + \frac{E_a^+}{E_a^+ + |\mathbf{p}| - g_{aa}^0} [e^{-2iE_a^+ t} \{ a_a^+(\mathbf{p}) b_a^+(\mathbf{p}) + b_a^+(\mathbf{p}) a_a^+(\mathbf{p}) \} \\
&\quad \left. \left. + e^{2iE_a^+ t} \{ [a_a^+(\mathbf{p})]^* [b_a^+(\mathbf{p})]^* + [b_a^+(\mathbf{p})]^* [a_a^+(\mathbf{p})]^* \} \right] \right\}, \tag{3.5}
\end{aligned}$$

in terms of the creation, $[a_a^{\pm}(\mathbf{p})]^*$ and $[b_a^{\pm}(\mathbf{p})]^*$, as well as the annihilation, $a_a^{\pm}(\mathbf{p})$ and $b_a^{\pm}(\mathbf{p})$, operators. Note that the operators $a_a^{\pm}(\mathbf{p})$ and $b_a^{\pm}(\mathbf{p})$ are independent up to now.

Let us establish the following relation:

$$a_a^{\pm}(\mathbf{p})(E_a^{\pm} + |\mathbf{p}| \mp g_{aa}^0) = 4b_a^{\pm}(\mathbf{p})(|\mathbf{p}| \mp g_{aa}^0), \tag{3.6}$$

and the analogous expression for conjugate operators, as well as suggest that the operators $a_a^{\pm}(\mathbf{p})$ obey the anti-commutation properties,

$$\{a_a^{\pm}(\mathbf{k}); [a_b^{\pm}(\mathbf{p})]^*\}_+ = \delta_{ab} \delta^3(\mathbf{k} - \mathbf{p}), \tag{3.7}$$

with all the rest of the anticommutators being equal to zero. In this case the time dependent terms in Eq. (3.5) are washed out. The remaining terms can be represented

as

$$\begin{aligned}
H_0 &= \int d^3\mathbf{p} \sum_a [E_a^- (a_a^-)^* a_a^- + E_a^+ (a_a^+)^* a_a^+] \\
&\quad + \text{divergent terms}, \tag{3.8}
\end{aligned}$$

which shows that the total energy of a massive Weyl field is a sum of energies corresponding to elementary oscillators of positive and negative helicities.

Using the results of Ref. [10] we can also quantize the total momentum of a Weyl field defined as

$$\begin{aligned}
\mathbf{P}_0 &= \int d^3\mathbf{r} \sum_a \left[\left(\eta_a^{(0)*} \right)^T \nabla \pi_a^{(0)*} \right. \\
&\quad \left. - \left(\pi_a^{(0)} \right)^T \nabla \eta_a^{(0)} \right]. \tag{3.9}
\end{aligned}$$

With help of Eqs. (3.3), (3.6), and (3.7) we rewrite Eq. (3.9) in the following form:

$$\mathbf{P}_0 = \int d^3\mathbf{p} \sum_a \mathbf{p} [(a_a^-)^* a_a^- + (a_a^+)^* a_a^+] + \text{divergent terms}, \quad (3.10)$$

which has the similar structure as Eq. (3.8).

Note that the divergent terms in Eqs. (3.8) and (3.10) contain the factor $\delta^3(\mathbf{p} = 0) \rightarrow \infty$, which can be formally removed by the normal ordering of the operators a_a^\pm and $(a_a^\pm)^*$. It is also interesting to mention that a massive Weyl field in vacuum can be quantized in the two independent ways (see Ref. [10]) because of the degeneracy of the neutrino energy levels: $E_a^- = E_a^+ = \sqrt{m_a^2 + |\mathbf{p}|^2}$. On the contrary, in matter only one of the possibilities for the quantization gives the correct result for the total energy (3.8) since the energy levels are no longer degenerate, cf. Eq. (3.4).

IV. NONDIAGONAL INTERACTION WITH MATTER AND ELECTROMAGNETIC FIELD

In this section we apply the approach for the quantization of a massive Weyl in a background matter developed in Sec. III for the treatment of the nondiagonal Hamiltonian H_{int} given in Eq. (3.2). On the basis of the obtained results and using the density matrix formalism [25] we derive the effective Hamiltonian for the description of neutrino spin-flavor oscillations in matter and electromagnetic field. Then we demonstrate that for ultrarelativistic particles our effective Hamiltonian is consistent with the analogous expression obtained in frames of the standard quantum mechanical approach.

To quantize the Hamiltonian H_{int} we shall use the forward scattering approximation. It means that one has to account for only the terms conserving the number of particles [26]. Using Eqs. (3.3), (3.6), and (3.7) we rewrite Eq. (3.2) in the form,

$$H_{\text{int}} = \int d^3\mathbf{p} \sum_{a \neq b} [M_{ab}^-(a_a^-)^* a_b^- e^{i\delta_{ab}^- t} + M_{ab}^+(a_a^+)^* a_b^+ e^{i\delta_{ab}^+ t} + F_{ab}^-(a_a^-)^* a_b^+ e^{i\sigma_{ab}^- t} + F_{ab}^+(a_a^+)^* a_b^- e^{i\sigma_{ab}^+ t}], \quad (4.1)$$

where $\delta_{ab}^\pm = E_a^\pm - E_b^\pm$, $\sigma_{ab}^\pm = E_a^\pm - E_b^\mp$,

$$\begin{aligned} M_{ab}^\pm = & \mp \frac{g_{ab}}{4} \left(\frac{E_a^\pm + |\mathbf{p}| \mp g_{aa}^0}{|\mathbf{p}| \mp g_{aa}^0} + \frac{E_b^\pm + |\mathbf{p}| \mp g_{bb}^0}{|\mathbf{p}| \mp g_{bb}^0} \right) \left[1 + \frac{m_a m_b}{(E_a^\pm + |\mathbf{p}| \mp g_{aa}^0)(E_b^\pm + |\mathbf{p}| \mp g_{bb}^0)} \right] \\ & \pm \frac{\mu_{ab}}{4} \left\{ \frac{1}{|\mathbf{p}| \mp g_{aa}^0} \left[m_a w_+^T(\boldsymbol{\sigma}[\mathbf{B} \mp i\mathbf{E}]) w_+^* + m_b w_-^T(\boldsymbol{\sigma}[\mathbf{B} \pm i\mathbf{E}]) w_-^* \frac{E_a^\pm + |\mathbf{p}| \mp g_{aa}^0}{E_b^\pm + |\mathbf{p}| \mp g_{bb}^0} \right] \right. \\ & \left. + \frac{1}{|\mathbf{p}| \mp g_{bb}^0} \left[m_b w_+^T(\boldsymbol{\sigma}[\mathbf{B} \pm i\mathbf{E}]) w_+^* + m_a w_-^T(\boldsymbol{\sigma}[\mathbf{B} \mp i\mathbf{E}]) w_-^* \frac{E_b^\pm + |\mathbf{p}| \mp g_{bb}^0}{E_a^\pm + |\mathbf{p}| \mp g_{aa}^0} \right] \right\}, \end{aligned} \quad (4.2)$$

and

$$\begin{aligned} F_{ab}^\pm = & \frac{\mu_{ab}}{4} \left\{ w_\pm^T(\boldsymbol{\sigma}[\mathbf{B} \pm i\mathbf{E}]) w_\mp^* \left[\frac{E_a^\pm + |\mathbf{p}| \mp g_{aa}^0}{|\mathbf{p}| \mp g_{aa}^0} + \frac{E_b^\mp + |\mathbf{p}| \pm g_{bb}^0}{|\mathbf{p}| \pm g_{bb}^0} \right] \right. \\ & \left. - m_a m_b w_\mp^T(\boldsymbol{\sigma}[\mathbf{B} \mp i\mathbf{E}]) w_\pm^* \left[\frac{1}{(E_a^\pm + |\mathbf{p}| \mp g_{aa}^0)(|\mathbf{p}| \pm g_{bb}^0)} + \frac{1}{(E_b^\mp + |\mathbf{p}| \pm g_{bb}^0)(|\mathbf{p}| \mp g_{aa}^0)} \right] \right\}. \end{aligned} \quad (4.3)$$

Note that Eqs. (4.1)-(4.3) are valid for arbitrary neutrino masses, momentum, and the diagonal neutrino interaction with matter.

Now we define the neutrino density matrix as

$$\delta^3(\mathbf{p} - \mathbf{k}) \rho_{AB}(\mathbf{p}) = \langle a_B^*(\mathbf{p}) a_A(\mathbf{k}) \rangle, \quad (4.4)$$

where $A = (\zeta, a)$ is a composite index and $\langle \dots \rangle$ is the statistical averaging over the neutrino ensemble. In principle, we could interchange the indexes A and B in the rhs of Eq. (4.4). Since we study the dynamics of Majorana neutrinos, such a transposition would mean the consideration of neutrinos as antiparticles rather than as particles

as we do here. Anyway both definitions will give equivalent results. Note that a density matrix, having both neutrino type and helicity indexes, which is analogous to Eq. (4.4), was studied in Ref. [27] where the interaction between Dirac neutrinos, mediated by a scalar boson, was discussed.

Applying the quantum Liouville equation for the description of the density matrix evolution,

$$i\dot{\rho} = [\rho, H_{\text{int}}], \quad (4.5)$$

we can rewrite it as $i\dot{\rho} = [\mathcal{H}, \rho]$ using the effective quan-

tum mechanical Hamiltonian,

$$\mathcal{H} = \begin{pmatrix} (M_{ab}^- e^{i\delta_{ab}^- t}) & (F_{ab}^- e^{i\sigma_{ab}^- t}) \\ (F_{ab}^+ e^{i\sigma_{ab}^+ t}) & (M_{ab}^+ e^{i\delta_{ab}^+ t}) \end{pmatrix}. \quad (4.6)$$

Again we stress that yet no expansion over the parameter $m_a/|\mathbf{p}|$, which is small for ultrarelativistic neutrinos, is made. Thus Eqs. (4.5) and (4.6) are valid for the description of neutrinos with arbitrary initial momentum.

To study the evolution of our system we use Eq. (3.3), where the wave functions already contain time dependent exponential factors. The energies in Eq. (3.3) correspond to the total diagonal Hamiltonian (3.1), cf. Eqs. (3.4) and (3.8), which contains both the mass term and the diagonal interaction with a background matter rather than only a kinetic term as in Ref. [25]. Thus our treatment is analogous to the Dirac picture of the quantum theory. That is why in Eq. (4.5) it is sufficient to commute the density matrix only with H_{int} rather than with the total Hamiltonian $H = H_0 + H_{\text{int}}$.

To derive Eqs. (4.1)-(4.3) we suppose that the external fields g_{ab}^0 , $a \neq b$, and $F_{\mu\nu}$ are spatially constant. In fact, for this supposition to be valid, the characteristic scale of the external field variation L_{ext} should be much bigger

than the typical width of the neutrino wave packet \hbar/E_ν . One can see that in almost all realistic situations external fields can be regarded as constant for the derivation of an effective Hamiltonian. Nevertheless, when the obtained effective Hamiltonian is used to describe the dynamics of the system, e.g., with help of Eq. (4.5), we should account for the variation of external fields.

To demonstrate the consistency of the obtained results with the achievements of the standard quantum mechanical description of neutrino spin-flavor oscillations (see, e.g., Ref. [2]) we discuss the simplest case of the two neutrino eigenstates, $a = 1, 2$, and consider the situation of ultrarelativistic particles, $|\mathbf{k}| \gg \max(m_a, g_{aa}^0)$, where \mathbf{k} is the initial momentum of neutrinos. In this case we should decompose the energy levels (3.4) as

$$E_a^\pm = |\mathbf{k}| + \frac{m_a^2}{2|\mathbf{k}|} \mp g_{aa}^0 + \dots \quad (4.7)$$

Then, supposing that $\mathbf{E} = 0$, since it is difficult to create a large scale electric field, we get that $M_{ab}^\pm \approx \mp g_{ab}$ and $F_{ab}^\pm \approx -\mu_{ab}|\mathbf{B}| \sin \vartheta_{\mathbf{kB}}$, where $\vartheta_{\mathbf{kB}}$ is the angle between the vectors \mathbf{k} and \mathbf{B} .

To eliminate the time dependent factors in the effective Hamiltonian (4.6) we make the transformation of the density matrix [17],

$$\rho_{\text{qm}} = \mathcal{U} \rho \mathcal{U}^\dagger, \quad \mathcal{U} = \text{diag}\{e^{-i(\Phi+g_{11}^0)t}, e^{i(\Phi-g_{22}^0)t}, e^{-i(\Phi-g_{11}^0)t}, e^{i(\Phi+g_{22}^0)t}\}, \quad (4.8)$$

where $\Phi = \delta m^2/4|\mathbf{k}|$ is the phase of vacuum oscillations and $\delta m^2 = m_1^2 - m_2^2$ is the mass squared difference.

The evolution of the transformed density matrix can be represented as $i\dot{\rho}_{\text{qm}} = [\mathcal{H}_{\text{qm}}, \rho_{\text{qm}}]$, where the new effective Hamiltonian has the form,

$$\mathcal{H}_{\text{qm}} = \mathcal{U} \mathcal{H} \mathcal{U}^\dagger + i\dot{\mathcal{U}} \mathcal{U}^\dagger = \begin{pmatrix} \Phi + g_{11}^0 & g_{12}^0 & 0 & -\mu_{12}|\mathbf{B}| \sin \vartheta_{\mathbf{kB}} \\ g_{21}^0 & -\Phi + g_{22}^0 & -\mu_{21}|\mathbf{B}| \sin \vartheta_{\mathbf{kB}} & 0 \\ 0 & -\mu_{12}|\mathbf{B}| \sin \vartheta_{\mathbf{kB}} & \Phi - g_{11}^0 & -g_{12}^0 \\ -\mu_{21}|\mathbf{B}| \sin \vartheta_{\mathbf{kB}} & 0 & -g_{21}^0 & -\Phi - g_{22}^0 \end{pmatrix}. \quad (4.9)$$

Recalling the properties of the magnetic moments matrix for Majorana neutrinos, $\mu_{12} = i\mu$ and $\mu_{21} = -i\mu$, with μ being a real number, we can see that Eq. (4.9) reproduces the well known quantum mechanical Hamiltonian for spin-flavor oscillations of Majorana neutrinos in matter and a magnetic field [2].

Note that previously calculated transition probabilities of neutrino oscillations in a magnetic field [20] and in a background matter [21] can be re-derived using the effective Hamiltonian (4.9) which was obtained in frames of the approach involving canonically quantized Weyl fields. However, as we mentioned in Sec. II, methodologically our method for the description of massive Majorana neutrinos in external fields is more logical since it is based on the first principles of the quantum field theory.

V. SELF-INTERACTION

In this section we generalize the results of Secs. II-IV to include the neutrino self-interaction. First we reformulate the previously proposed Hamiltonian for the self-interaction in terms of the two-component Weyl fields and then we quantize it. Finally, using the density matrix formalism we derive the corresponding contribution to the quantum mechanical effective Hamiltonian and compare it with the previously obtained results.

The Hamiltonian describing the neutrino self-interaction, mediated by a neutral Z -boson, was derived in Refs. [6, 25] and has the form,

$$H_S = \int d^3\mathbf{r} \sum_{abcd} G_{ab} G_{cd} \bar{\psi}_a \gamma_\mu^L \psi_b \cdot \bar{\psi}_c \gamma_\mu^L \psi_d, \quad (5.1)$$

where $\gamma_\mu^L = \gamma_\mu(1 - \gamma^5)/2$ and G_{ab} are the coefficients which depend on the neutrino interactions channel. The explicit form of these coefficients can be found in

Ref. [28]. Let us choose the wave function ψ_a in Eq. (5.1) as $\psi_a^{(\eta)}$ in Eq. (2.2). Therefore we express the Hamiltonian (5.1) in terms of the two-component Weyl spinors as

$$H_S = \int d^3\mathbf{r} \sum_{abcd} G_{ab} G_{cd} \eta_a^\dagger \sigma_\mu \eta_b \cdot \eta_c^\dagger \sigma^\mu \eta_d, \quad (5.2)$$

where $\sigma_\mu = (1, \boldsymbol{\sigma})$. It should be noted that we presented the heuristic derivation of Eq. (5.2) from Eq. (5.1). In principle we could just postulate Eq. (5.2).

Using Eqs. (3.3) and (3.6) we cast the self-interaction Hamiltonian (5.2) into the form,

$$H_S = \frac{1}{(2\pi)^3} \int d^3\mathbf{p} d^3\mathbf{p}' d^3\mathbf{q} d^3\mathbf{q}' \delta(\mathbf{p} + \mathbf{q} - \mathbf{p}' - \mathbf{q}') \sum_{abcd} G_{ab} G_{cd} [\eta_a(\mathbf{q})^*]^T \sigma_\mu \eta_b(\mathbf{q}') \cdot [\eta_c(\mathbf{p})^*]^T \sigma^\mu \eta_d(\mathbf{p}'), \quad (5.3)$$

where

$$\eta_a(\mathbf{p}) = \sum_{\zeta=\pm 1} \left[a_a^{(\zeta)}(\mathbf{p}) u_a^{(\zeta)}(\mathbf{p}) e^{-iE_a^{(\zeta)} t} + \left[a_a^{(\zeta)}(-\mathbf{p}) \right]^* v_a^{(\zeta)}(-\mathbf{p}) e^{iE_a^{(\zeta)} t} \right], \quad (5.4)$$

is the Fourier transform of the wave function η_a and

$$u_a^-(\mathbf{p}) = w_-(\mathbf{p}), \quad v_a^+(\mathbf{p}) = w_-(\mathbf{p}), \quad u_a^+(\mathbf{p}) = -\frac{m_a}{E_a^+ + |\mathbf{p}| - g_{aa}^0} w_+(\mathbf{p}), \quad v_a^-(\mathbf{p}) = \frac{m_a}{E_a^+ + |\mathbf{p}| + g_{aa}^0} w_+(\mathbf{p}), \quad (5.5)$$

are the basis spinors rewritten in a formalized manner, cf. Eq. (3.3).

Applying Eq. (4.5) to account for the contribution of the self-interaction to the dynamics of the system and again working in the forward scattering limit [26], we can represent the evolution of the density matrix, $i\dot{\rho} = [\mathcal{H}_S, \rho]$, using the effective Hamiltonian \mathcal{H}_S . After a bit lengthy but straightforward calculations we get the following expression for \mathcal{H}_S :

$$\begin{aligned} \mathcal{H}_S = & 2 \int \frac{d^3\mathbf{p}}{(2\pi)^3} \{ M_\mu(\mathbf{k}, \mathbf{p}) \text{tr}\{[M^\mu(\mathbf{p}, \mathbf{p}) - N^\mu(\mathbf{p}, \mathbf{p})]\rho(\mathbf{p})\} + N_\mu(\mathbf{k}, \mathbf{p}) \text{tr}\{[N^\mu(\mathbf{p}, \mathbf{p}) - M^\mu(\mathbf{p}, \mathbf{p})]\rho(\mathbf{p})\} \\ & - [M_\mu(\mathbf{k}, \mathbf{p}) - N_\mu(\mathbf{k}, \mathbf{p})]\rho(\mathbf{p})[M^\mu(\mathbf{p}, \mathbf{k}) - N^\mu(\mathbf{p}, \mathbf{k})] + (K_\mu(\mathbf{k}, \mathbf{p}) - [K_\mu(\mathbf{k}, \mathbf{p})]^T)\rho^T(\mathbf{p})(L^\mu(\mathbf{p}, \mathbf{k}) - [L^\mu(\mathbf{p}, \mathbf{k})]^T) \}, \end{aligned} \quad (5.6)$$

where

$$\begin{aligned} M_{AB}^\mu(\mathbf{p}, \mathbf{k}) &= G_{ab} e^{iE_A(\mathbf{p})t} \langle u_A(\mathbf{p}) | \sigma^\mu | u_B(\mathbf{k}) \rangle e^{-iE_B(\mathbf{k})t}, & K_{AB}^\mu(\mathbf{p}, \mathbf{k}) &= G_{ab} e^{iE_A(\mathbf{p})t} \langle u_A(\mathbf{p}) | \sigma^\mu | v_B(\mathbf{k}) \rangle e^{iE_B(\mathbf{k})t}, \\ N_{AB}^\mu(\mathbf{p}, \mathbf{k}) &= G_{ba} e^{-iE_B(\mathbf{k})t} \langle v_B(\mathbf{k}) | \sigma^\mu | u_A(\mathbf{p}) \rangle e^{iE_A(\mathbf{p})t}, & L_{AB}^\mu(\mathbf{p}, \mathbf{k}) &= G_{ab} e^{-iE_A(\mathbf{p})t} \langle v_A(\mathbf{p}) | \sigma^\mu | u_B(\mathbf{k}) \rangle e^{-iE_B(\mathbf{k})t}. \end{aligned} \quad (5.7)$$

In Eq. (5.6) the transposition means the interchange of both discrete and continuous indexes, i.e. $[L_{AB}^\mu(\mathbf{p}, \mathbf{k})]^T = G_{ba} e^{-iE_B(\mathbf{k})t} \langle v_B(\mathbf{k}) | \sigma^\mu | u_A(\mathbf{p}) \rangle e^{-iE_A(\mathbf{p})t}$ etc.

Note that Eqs. (5.6) and (5.7) are valid for arbitrary neutrino masses, initial momenta, and the diagonal neutrino interaction with matter. However the analysis of these expressions is quite cumbersome. That is why again we discuss the situation of the two neutrino generations, $a = 1, 2$, and suppose that neutrinos are ultrarelativistic particles, $|\mathbf{k}| \gg \max(m_a, g_{aa}^0)$. Then, to eliminate the time dependence in Eq. (5.7) we make the additional matrix transformation of the effective Hamiltonian, $\mathcal{H}_{\text{qm}} = \mathcal{U} \mathcal{H}_S \mathcal{U}^\dagger$, where $\mathcal{H}_S = \mathcal{H}_S[\mathcal{U} \rho \mathcal{U}^\dagger]$, since \mathcal{H}_S is the function of ρ , and the matrix \mathcal{U} is defined in Eq. (4.8).

Finally we can represent the contribution of the neutrino self-interaction to the quantum mechanical effective Hamiltonian as,

$$\mathcal{H}_{\text{qm}}(\mathbf{k}) = \text{diag}(\mathcal{H}_{--}, \mathcal{H}_{++}), \quad (5.8)$$

where

$$\mathcal{H}_{--} = 2 \int \frac{d^3\mathbf{p}}{(2\pi)^3} (1 - \cos \vartheta_{\mathbf{k}\mathbf{p}}) \{ G \text{tr}[G \rho_{--}(\mathbf{p}) - G^T \rho_{++}(\mathbf{p})] + G[\rho_{--}(\mathbf{p}) - \rho_{++}^T(\mathbf{p})] G \},$$

$$\mathcal{H}_{++} = 2 \int \frac{d^3 \mathbf{p}}{(2\pi)^3} (1 - \cos \vartheta_{\mathbf{k}\mathbf{p}}) \{ G^T \text{tr}[G^T \rho_{++}(\mathbf{p}) - G \rho_{--}(\mathbf{p})] + G^T [\rho_{++}(\mathbf{p}) - \rho_{--}^T(\mathbf{p})] G^T \}, \quad (5.9)$$

and $\mathcal{H}_{\pm\mp} = 0$. Here $\vartheta_{\mathbf{k}\mathbf{p}}$ is the angle between the vectors \mathbf{k} and \mathbf{p} and we use the helicity components of the density matrix,

$$\rho_{\text{qm}} = \begin{pmatrix} \rho_{--} & \rho_{-+} \\ \rho_{+-} & \rho_{++} \end{pmatrix}. \quad (5.10)$$

To derive Eqs. (5.8) and (5.9) we use the identity for two-component c -number spinors, $\eta_1^\dagger \sigma_\mu \eta_2 \cdot \eta_3^\dagger \sigma^\mu \eta_4 = -\eta_1^\dagger \sigma_\mu \eta_4 \cdot \eta_2^\dagger \sigma^\mu \eta_3$, which results from the Fierz transformation of four-component spinors, $\bar{\psi}_1 \gamma_\mu (1 - \gamma^5) \psi_2 \cdot \bar{\psi}_3 \gamma^\mu (1 - \gamma^5) \psi_4 = -\bar{\psi}_1 \gamma_\mu (1 - \gamma^5) \psi_4 \cdot \bar{\psi}_2 \gamma^\mu (1 - \gamma^5) \psi_3$.

One can conclude from Eq. (5.9) that the self-interaction influences spin-flavor oscillations of neutrinos. However, since the nondiagonal terms in Eq. (5.8) are equal to zero for ultrarelativistic particles, the self-interaction cannot directly induce transitions between different helicity states.

It should be noted that in the majority of works where collective effects in neutrino oscillations were studied, the case of Dirac neutrinos was examined. Although one can expect that in the ultrarelativistic case the dynamics of Dirac and Majorana neutrinos should be similar, we cannot reach a complete coincidence because Dirac particles have twice more degrees of freedom, i.e. an additional density matrix for antineutrinos is required. Nevertheless let us check the consistency of our findings with the previously obtained results. First we should chose a definite helicity. For example, we may put $\mathcal{H}_{--} \neq 0$ and $\mathcal{H}_{++} = 0$. Then, defining the “antineutrino” density matrix as $\bar{\rho} = \rho_{++}^T$, cf. Eq. (4.4), we re-derive the contribution of the neutrino self-interaction to the effective Hamiltonian obtained in Ref. [25].

At the end of this section we notice that the presented derivation of Eqs. (5.8) and (5.9) is not unique. In the general self-interaction Hamiltonian (5.1) we can set $\psi_a = \psi_a^{(\xi)}$, where $\psi_a^{(\xi)}$ is defined in Eq. (2.2). Thus the Hamiltonian H_S can be expressed in terms of the canonical momenta,

$$H_S = \int d^3 \mathbf{r} \sum_{abcd} G_{ab} G_{cd} \pi_a^T \sigma_\mu \pi_b^* \cdot \pi_c^T \sigma^\mu \pi_d^*, \quad (5.11)$$

where we use the relation between ξ_a and π_a : $\xi_a = i\sigma_2 \pi_a$ (see also Sec. II).

Then we should re-define the wave functions η_a and ξ_a in Eq. (3.3), introducing the additional multiplier $1/2$ in each spinor, as well as the connection between operators $a_a^\pm(\mathbf{p})$ and $b_a^\pm(\mathbf{p})$, which now reads, $a_a^\pm(\mathbf{p})(E_a^\pm + |\mathbf{p}| \mp g_{aa}^0) = b_a^\pm(\mathbf{p})(|\mathbf{p}| \mp g_{aa}^0)$, cf. Eq. (3.6). Note that these modifications do not affect the results of Secs. III and IV. Performing the same calculations which led us to Eqs. (5.8) and (5.9), but using the modified Hamiltonian (5.11), we get that for ultrarelativistic particles

the contribution of the neutrino self-interaction to the quantum mechanical effective Hamiltonian coincides with Eqs. (5.8) and (5.9).

VI. CONCLUSION

In summary we mention that in the present work we have constructed the consistent quantum theory of a system of massive Weyl fields propagating in a background matter and interacting between themselves and with an external electromagnetic field. We have obtained several important results.

First, in Sec. II, the classical field theory description of a massive Weyl field in an arbitrarily moving and polarized matter and an electromagnetic field has been presented. Using the approach developed in Ref. [10], where the evolution of a massive Weyl field in vacuum was studied, we have derived the classical Hamiltonian (2.5) for our system. Then applying the canonical Hamilton equations (2.6) and (2.7) we have re-obtained the analog of the well known wave equation (2.1) for a Majorana neutrino in matter and an electromagnetic field. This our result corrects the previous statement [20] that massive Majorana particles are essentially quantum objects described only using the creation and annihilation operators formalism. Moreover, now we have expanded the classical field theory approach, cf. Ref. [10], to include the interaction with matter and an electromagnetic field.

Second, in Sec. III, we have canonically quantized massive Weyl fields in a non-moving and unpolarized matter. We have used the plane wave solution (3.3) and (3.4) (see also Ref. [17]) of the corresponding wave equation, where we supposed that the expansion coefficients are the operators. Then, requiring the positive definiteness of the total energy (3.5), we have obtained that the operator expansion coefficients should satisfy the anticommutation properties (3.7). It is interesting to mention that unlike the quantization of a Weyl field in vacuum, where two independent quantization schemes are possible, in matter there is only one opportunity (3.6), which gives the correct form for the total energy (3.8) and the total momentum (3.10).

Third, in Sec. IV, we have applied the elaborated quantization method for the nondiagonal interaction with matter and an electromagnetic field (3.2). Within the forward scattering approximation we have derived the quantized interaction Hamiltonian (4.1)-(4.3), which is valid for arbitrary neutrino masses, an initial momentum, and the diagonal neutrino interaction with matter. Then, using the density matrix formalism, developed in Ref. [25],

and in the approximation of ultrarelativistic particles we have re-derived the effective Hamiltonian (4.9), previously obtained in frames of the standard quantum mechanical approach [2], for the description of spin-flavor oscillations of Majorana neutrinos in matter and a magnetic field.

Finally, in Sec. V, we have quantized the self-interaction (5.1) of Majorana neutrinos using the developed formalism. Again in the forward scattering approximation we have got the contribution to the effective Hamiltonian (5.6) and (5.7) which is valid for neutrinos with arbitrary masses and an initial momentum. Then, for ultrarelativistic particles we have compared our results with the previously obtained effective Hamiltonian [25], which describes collective neutrino oscillations, and have found the consistency.

Note that in all the previous works where collective effects in neutrino flavor oscillations were studied the case of Dirac neutrinos in a background matter was considered. Thus in the present work for the first time we have discussed the situation of Majorana neutrinos and generalized the consideration to include an external magnetic field. It should be noted that for supernova neutrinos both the interaction with a dense background matter, a strong magnetic field, and the neutrino self-interaction can be of equal importance. Therefore, the effective Hamiltonians (4.9), (5.8), and (5.9), derived in our work, may be used for the treatment of collective effects in spin-

flavor oscillations of supernova neutrinos in matter and a magnetic field.

The wave functions (3.3) exactly take into account the diagonal neutrino interaction with background matter, g_{aa}^0 , whereas in Sec. IV the external fields, g_{ab}^0 , $a \neq b$, and $F_{\mu\nu}$ are treated perturbatively, with only linear term being kept, cf. Eqs. (4.2) and (4.3). However in general case the potentials g_{aa}^0 and g_{ab}^0 can be of the same order of magnitude. Nevertheless, besides supernova neutrinos discussed here, we may apply the obtained results to study nonperturbative effects in neutrino oscillations if the nondiagonal potential of matter interaction g_{ab}^0 is small or negligible. For instance, such a situation is implemented when neutrinos propagate in the inner crust of a neutron star, where $n_{e,p} \ll n_n$, with $n_{e,p,n}$ being the number densities of electrons, protons, and neutrons respectively. Another example, when the nondiagonal matter interaction is unimportant, is $\nu_\mu \leftrightarrow \nu_\tau$ oscillations channel.

ACKNOWLEDGMENTS

I am very thankful to A.E. Lobanov, G.G. Raffelt, and V.B. Semikoz for helpful discussions as well as to FAPESP (Brazil) for a grant.

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